Indian Institute of Management Calcutta

Working Paper Series

WPS No. 787
September 2016

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Improving DEA efficiency under constant sum of inputs/outputs and common weights

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Abstract

In Data Envelopment Analysis (DEA), one way of calculating efficiency is to use parameter weights common to all decision-making units (DMUs), since it is reasonable for similar DMUs to place similar weights on inputs and outputs. At the same time, in many situations, the total amount of input or output available to a set of DMUs is fixed. In this paper, we have formulated DEA models to calculate the best strategy for improving the efficiency of an inefficient DMU when the parameters of all DMUs are weighted with a common set of weights, and there is a constant sum of inputs/outputs constraint. Theoretical results have been illustrated with the help of a numerical example.

Keywords: DEA, efficiency, linear programming, common weights, constant sum of inputs, constant sum of outputs

1 Introduction

Data Envelopment Analysis (DEA) is a widely applied non-parametric mathematical programming technique to calculate the relative efficiency of firms/organizations or Decision Making Units (DMUs) operating in a similar environment and utilizing multiple inputs to produce multiple outputs. Based on Farrell's (1957) work on productive efficiency, DEA was first introduced by Charnes, Cooper, and Rhodes (1978). Efficiency obtained using DEA, in its simplest form, is the comparison of the weighted output to weighted input ratio of the observing DMU with that of the best practice in the group. DEA has an advantage over other methods since the inputs/output weights are determined by the DEA model itself, and thus the decision maker does not face the problem of determining the weights beforehand. Measurement of efficiency is important to shareholders, managers, and investors for any future course of action. DEA has been extensively applied to a wide spectrum of practical problems. Examples include financial institutions (Sherman and Ladino 1995), bank failure prediction (Barr et. al. 1994), electric utilities evaluation (Goto and Tsutsui 1998), textile industry performance (Chandra et. al. 1998), sports (Singh and Adhikari, 2015; Singh 2011; Ruiz et al. 2013), portfolio evaluation (Murthi et. al. 1997). DEA analysis makes allowance for DMUs which are under constant returns to scale (CRS) using the Charnes-Cooper-Rhodes (CCR) models (1978) as well as for variable returns to

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scale (VRS) models such as the Banker-Charnes-Cooper (BCC) models (Banker et al. 1984).

A DMU can improve its efficiency by reducing its input or increasing its output or doing both. However, when a DMU attempts to increase output, it may turn out that there is only a limited amount of output that can be produced in the system by all DMUs, such as the Olympic Games where only a limited number of medals are available to the competing teams (Lins et. al. 2003), or competition for market share (Hu and Fang 2010). This limitation on output is called Constant Sum of Outputs (CSOO). Yang et. al. (2011) established models for improving the efficiency of a DMU under CSOO, but these models all operated under the assumption that the various inputs and outputs of a DMU were freely substitutable, and that the weights of one DMU were independent of the weights for other DMUs.

Similarly, if the amount of input in the system is constant, a DMU can only reduce its input if the input of another DMU is increased. Examples of constant sum of input (CSOI) problems include the distribution of a fixed cost (Cook and Kress 1999) or the allocation of office space (Gomes et al. 2008). Almost all approaches to the CSOI problem such as those developed by Beasley (2003) or later methods like the ellipsoidal frontier model (Milioni et al. 2011), treat the CSOI as a fixed cost problem. Lotfi et al. (2013) deals with resource allocation under common weights, but it also treats the problem as a fixed cost allocation. None of them consider the problem of a single DMU attempting to improve its efficiency, instead looking at all DMUs simultaneously. Furthermore, no paper exists that combines CSOI and CSOO constraints in a single problem.

The survey of existing works in CSOO and CSOI indicate that there are few papers that take into account the Common Weights restriction, and none of these addresses both CSOO and CSOI problems simultaneously. Common Weights (CW) is a very common form of weight restriction in DEA. If the DMUs are operating in a similar environment, they can be expected to have the same parameter weights (Lotfi et al. 2013). The CW approach is also used when classic DEA models do not give us accurate information as to the real weights of the input and output parameters, or to differentiate between efficient DMUs (Liu and Peng 2008). Several approaches exist for solving CW efficiency (Roll and Golany 1993, Cook and Zhu 2007). In this paper, we have addressed the problem of efficiency improvement under CSOI and CSOO constraints when a set of common weights is applied to input and output parameters of all DMUs. This is useful in real-life situations where the overall efficiency of the system is more important than individual DMU's efficiency (Beasley 2003), and there are limits to total productivity and/or a fixed input quantity. It is also useful in situations where all DMUs are expected to place similar weights on the various parameters.

This paper is organized as follows. In section 2, DEA models and theoretical results have been developed. Numerical illustration of the theoretical results developed in this paper is given in section 3. Conclusion and future direction for research have been presented in section 4.

2 Model formulation and theoretical results

The following notations have been used throughout this paper. Other notations used in certain sections will be defined at appropriate places.

n : The number of DMUs.
m : The number of inputs.
s : The number of outputs.

x_{ij} : Observed amount of j^{th} input for the i^{th} DMU $(i = 1, \ldots, n; j = 1, \ldots, m)$.
y_{ir} : Observed amount of r^{th} output for the i^{th} DMU$(i = 1, \ldots, n; r = 1, \ldots, s)$.
θ_k^{CW} : Efficiency of the k^{th} DMU under common weights (CW).
u_r : The weight assigned to the r^{th} output$(r = 1, \ldots, s)$.
v_j : The weight assigned to the j^{th} input$(j = 1, \ldots, m)$.
u_0 : Value representing the variable part of variable returns to scale DEA models.
Δ_i^I : Virtual gap between the weighted inputs of the $i^{th}(i = 1, \ldots, n)$ DMU and best-practices frontier.
Δ_i^O : Virtual gap between the weighted outputs of the $i^{th}(i = 1, \ldots, n)$ DMU and best-practices frontier.
g_{kr} : The output increase in the $r^{th}(r = 1, \ldots, s)$ output of the k^{th} DMU.
t_{ir} : The output reduction in the $r^{th}(r = 1, \ldots, s)$ output of the $i^{th}(i = 1, \ldots, n; i \neq k)$ DMU.
f_{kj} : The input decrease in the $j^{th}(j = 1, \ldots, m)$ input of the k^{th} DMU.
s_{ij} : The input increase in the $j^{th}(j = 1, \ldots, m)$ input of the $i^{th}(i = 1, \ldots, n; i \neq k)$ DMU.
ϵ : An infinitesimally small positive value.

Several methods exist for ranking DMUs' efficiency under common weights (Roll and Golany 1993, Cook and Zhu 2007, Liu and Peng 2008, Lotfi et al. 2013). Under the CW limitation, since the weights are same across all DMUs, the objective is to select weights such that the overall efficiency of all DMUs is maximized. Later papers use a Goal Programming (Tamiz et al. 1998) approach to convert the problem into a solvable linear programming problem. In the approach used by Lotfi et al. (2013), in order to maximize the overall efficiency, the total virtual gap is to be minimized. The virtual output gap Δ_i^O is defined as the gap between the ideal weighted sum of outputs on the efficiency frontier, and the real weighted sum of outputs. The virtual input gap Δ_i^I is the gap between the ideal weighted sum of inputs and the real weighted sum of inputs. Thus, if $y_{ir}^*(r = 1, \ldots, s)$ and $x_{ij}^*(j = 1, \ldots, m)$ are the ideal outputs and inputs on the efficiency frontier for the i^{th} DMU, then

$$\Delta_i^O = \sum_{r=1}^{s} u_r y_{ir}^* - \sum_{r=1}^{s} u_r y_{ir},$$

$$\Delta_i^I = \sum_{j=1}^{m} v_j x_{ij} - \sum_{j=1}^{m} v_j x_{ij}^*,$$

$$\Delta_i^O, \Delta_i^I \geq 0,$$

$$\frac{\sum_{r=1}^{s} u_r y_{ir}^*}{\sum_{j=1}^{m} v_j x_{ij}^*} = 1.$$

Minimizing the total virtual gap across all DMUs means minimizing the value $\sum_{i=1}^{n}(\Delta_i^O + \Delta_i^I)$. The model to achieve this minimization is shown below.

$$(M1)\quad Min \sum_{i=1}^{n}(\Delta_i^O + \Delta_i^I)$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{ir} + \Delta_i^O}{\sum_{j=1}^{m} v_j x_{ij} - \Delta_i^I} = 1, i = 1, \ldots, n,$$

$$v_j,\ u_r, \Delta_i^O, \Delta_i^I \geq 0;\ i = 1, \ldots,\ n;\ j = 1, \ldots,\ m;\ r = 1, \ldots,\ s.$$

Model $(M1)$ is a nonlinear programming model. Lotfi et al. (2013) converted it to the following linear programming (LP) model:

$$(M2)\quad Min \sum_{i=1}^{n}(\Delta_i^O + \Delta_i^I)$$

subject to

$$\sum_{r=1}^{s} u_r y_{ir} - \sum_{j=1}^{m} v_j x_{ij} + (\Delta_i^O + \Delta_i^I) = 1, i = 1, \ldots, n,$$

$$v_j,\ u_r, \Delta_i^O, \Delta_i^I \geq 0;\ i = 1, \ldots,\ n;\ j = 1, \ldots,\ m;\ r = 1, \ldots,\ s.$$

In model $(M2)$, the higher the value of $(\Delta_i^O + \Delta_i^I)$, the lower the efficiency of the i^{th} DMU, and the i^{th} DMU is CW-efficient iff $(\Delta_i^O + \Delta_i^I) = 0$. If $u_r^*, v_j^*, \Delta_i^{O*}, \Delta_i^{I*}$ are the solutions to $(M1)$ then the efficiency ratio of any k^{th} DMU can be calculated as $\theta_k^{CW} = \frac{\sum_{r=1}^{s} u_r^* y_{kr}}{\sum_{j=1}^{m} v_j^* x_{kj}}$.

Without any loss of generality, it can be assumed that the first c input parameters are under CSOI constraint. This means that the sum of all changes in these parameters must be 0. Thus, if the j^{th} input of the k^{th} DMU i.e. x_{kj} is decreased by a certain amount f_{kj} then the value of the j^{th} inputs of the other DMUs will have to be increased. Let $s_{ij}(i \neq k, i = 1, \ldots,\ n)$ be the amount by which the j^{th} input of the $i^{th}(i \neq k)$ DMU is increased, then $f_{kj} = \sum_{i=1, i\neq k}^{n} s_{ij},\ f_{kj} < x_{kj}$. Similarly, it can be assumed that the first d output parameters are under CSOO constraint. If the r^{th} output of the k^{th} DMU i.e. y_{kr} is increased by a certain amount g_{kr} then the value of the r^{th} outputs of the other DMUs will have to be decreased. Let $t_{ir}(i \neq k, i = 1, \ldots,\ n)$ be the amount by which the r^{th} output of the $i^{th}(i \neq k)$ DMU is decreased, then $g_{kr} = \sum_{i=1, i\neq k}^{n} t_{ir},\ t_{ir} < y_{ir}$.

The objective is to minimize the virtual gap $\sum_{i=1}^{n}(\Delta_i^O + \Delta_i^I)$ while making the k^{th} DMU efficient. Since the k^{th} DMU is becoming efficient, $\Delta_k^I, \Delta_k^O = 0$. Incorporating these changes into $(M1)$, we get the following model:

$$(M3) \quad Min \sum_{i=1, i\neq k}^{n} (\Delta_i^O + \Delta_i^I)$$

subject to

$$\frac{\sum_{r=1}^{d} u_r(y_{kr} + g_{kr}) + \sum_{r=d+1}^{s} u_r y_{kr}}{\sum_{j=1}^{c} v_j(x_{kj} - f_{kj}) + \sum_{j=c+1}^{m} v_j x_{kj}} = 1,$$

$$\frac{\sum_{r=1}^{d} u_r(y_{ir} - t_{ir}) + \sum_{r=d+1}^{s} u_r y_{ir} + \Delta_i^O}{\sum_{j=1}^{c} v_j(x_{ij} + s_{ij}) + \sum_{j=c+1}^{m} v_j x_{ij} - \Delta_i^I} = 1, i = 1, \ldots, n; i \neq k,$$

$$g_{kr} = \sum_{i=1, i\neq k}^{n} t_{ir},$$

$$f_{kj} = \sum_{i=1, i\neq k}^{n} s_{ij},$$

$$t_{ir} \leq y_{ir} - \epsilon,\ r = 1, \ldots, d,\ i = 1, \ldots, n;\ i \neq k,$$

$$f_{kj} \leq x_{kj} - \epsilon,\ j = 1, \ldots, c,$$

$$g_{kr},\ t_{ir},\ f_{kj},\ s_{ij},\ v_j,\ u_r, \Delta_i^O, \Delta_i^I, \epsilon \geq 0;\ i = 1, \ldots,\ n;\ j = 1, \ldots,\ m;\ r = 1, \ldots,\ s.$$

The solution for the model $(M3)$ will give us the input decrease and/or output increase necessary in k^{th} DMU for the k^{th} DMU to become efficient while maximizing the overall efficiency across all DMUs.

Theorem 1. The model $(M3)$ will always have a feasible solution.

Proof. Let n be the number of DMUs. Apply $(M2)$ to the data set, and let $\Delta_i^{O*}, \Delta_i^{I*} (i = 1, \ldots, n)$, $u_r^*(r = 1, \ldots, s)$, $v_j^*(j = 1, \ldots, m)$ be the solution to $(M2)$. Select any positive values for $f_{kj}(f_{kj} < x_{kj})$ and $g_{kr} < \sum_{i=1, i\neq k}^{n} y_{ir}$ such that

$$\sum_{r=1}^{d} u_r^* g_{kr} = \Delta_k^{O*} \tag{2.1}$$

$$\sum_{j=1}^{c} v_j^* f_{kj} = \Delta_k^{I*} \tag{2.2}$$

Select any positive values for $t_{ir}(t_{ir} < y_{ir})$ and s_{ij} such that

$$g_{kr} = \sum_{i=1, i\neq k}^{n} t_{ir} (r = 1, \ldots, d) \tag{2.3}$$

$$f_{kj} = \sum_{i=1, i\neq k}^{n} s_{ij} (j = 1, \ldots, c) \tag{2.4}$$

Now, define the values of Δ_i^O and Δ_i^I $(i \neq k)$ as

$$\Delta_i^O = \Delta_i^{O*} + \sum_{r=1}^{d} u_r^* t_{ir}, (i = 1, \ldots, n; i \neq k) \tag{2.5}$$

$$\Delta_i^I = \Delta_i^{I*} + \sum_{j=1}^{c} v_j^* s_{ij}, (i = 1, \ldots, n; i \neq k) \tag{2.6}$$

Since $(M2)$ has been solved, and it is equivalent to $(M1)$, all the constraints of $(M1)$ are fulfilled by its solution. Thus,

$$\frac{\sum_{r=1}^{s} u_r^* y_{ir} + \Delta_i^{O*}}{\sum_{j=1}^{m} v_j^* x_{ij} - \Delta_i^{I*}} = 1, i = 1, \ldots, n.$$

The above equation can be rewritten using eqns. (2.1)-(2.6) as

$$\frac{\sum_{r=1}^{s} u_r^* y_{kr} + \sum_{r=1}^{d} u_r^* g_{kr}}{\sum_{j=1}^{m} v_j^* x_{kj} - \sum_{j=1}^{c} v_j^* f_{kj}} = 1, \tag{2.7}$$

$$\frac{\sum_{r=1}^{s} u_r^* y_{ir} + \Delta_i^O - \sum_{r=1}^{d} u_r^* t_{ir}}{\sum_{j=1}^{m} v_j^* x_{ij} - \Delta_i^I + \sum_{j=1}^{c} v_j^* s_{ij}} = 1,\ i = 1, \ldots, n; i \neq k, \tag{2.8}$$

$$f_{kj} = \sum_{i=1, i\neq k}^{n} s_{ij},\ j = 1, \ldots, c, \tag{2.9}$$

$$g_{kr} = \sum_{i=1, i\neq k}^{n} t_{ir},\ r = 1, \ldots, d, \tag{2.10}$$

$$0 \leq t_{ir} \leq y_{ir} - \epsilon,\ r = 1, \ldots, d;\ i = 1, \ldots, n;\ i \neq k, \tag{2.11}$$

$$0 \leq f_{kj} \leq x_{kj} - \epsilon,\ j = 1, \ldots, c;\ i = 1, \ldots, n;\ i \neq k, \tag{2.12}$$

The equations (2.7)-(2.12) are identical to the constraints of model $(M3)$, and they all hold good as they are equivalent to the constraints of $(M1)$. Thus the values of u_r^*, v_j^*, t_{ir}, s_{ij}, Δ_i^O, Δ_i^I as defined by $(M2)$ and eqns. (2.1)-(2.6) represent a feasible solution to model $(M3)$. Hence the proof. ■

Let E^* be the overall efficiency of all DMUs as calculated by $(M2)$, and E be overall efficiency off all DMUs using the model $(M3)$.

Theorem 2. $E \geq E^*$ for all solutions of $(M3)$.

Proof. The values of u_r^*, v_j^*, t_{ir}, s_{ij}, Δ_i^O, Δ_i^I as defined in Theorem 1, represent a feasible solution to model $(M3)$. Thus, the objective function value of model $(M3)$

$$
\begin{aligned}
& \sum_{i=1, i\neq k}^{n} (\Delta_i^O + \Delta_i^I) \\
= & \sum_{i=1, i\neq k}^{n} \left(\Delta_i^{O*} + \sum_{r=1}^{d} u_r^* t_{ir} + \Delta_i^{I*} + \sum_{j=1}^{c} v_j^* s_{ij} \right) \\
= & \sum_{i=1}^{n} (\Delta_i^{O*} + \Delta_i^{I*}) \\
= & \text{ Objective function value of model } (M2).
\end{aligned}
$$

Since the overall efficiency of all DMUs is inversely linked to the objective function value (the lower the value the higher the overall efficiency), and there exists a feasible solution where the objective function value of $(M2)$ and $(M3)$ is the same, this proves that there is at least a feasible solution of $(M3)$ for which $E = E^*$. Thus, since $(M3)$ minimizes the objective function to maximize overall efficiency, $E \geq E^*$ for all solutions of $(M3)$. ■

As the model $(M3)$ is a non-linear fractional programming model, it can be converted into an equivalent LP model by taking $\Phi_{kr} = u_r g_{kr}$, $\phi_{ir} = u_r t_{ir}$, $\Gamma_{kj} = v_j f_{kj}$, $\gamma_{ij} = v_j s_{ij}$ and carrying out cross-multiplication on the constraints. We also set $\sum_{j=1}^{m} v_j x_{kj} = 1$ to prevent the parameter weights from reaching extreme values. The LP model $(M4)$ is shown below.

$$
\begin{aligned}
(M4) \quad & Min \sum_{i=1, i\neq k}^{n} (\Delta_i^O + \Delta_i^I) \\
& \text{subject to} \\
& \sum_{r=1}^{s} u_r y_{kr} - \sum_{j=1}^{m} v_j x_{kj} + \sum_{r=1}^{d} \Phi_{kr} + \sum_{j=1}^{c} \Gamma_{kj} \ = \ 0, \\
& \sum_{r=1}^{s} u_r y_{ir} - \sum_{j=1}^{m} v_j x_{ij} + \Delta_i^O + \Delta_i^I - \sum_{r=1}^{d} \phi_{ir} - \sum_{j=1}^{c} \gamma_{kj} \ = \ 0, \ i = 1, \ldots, n; \ i \neq k, \\
& \sum_{j=1}^{m} v_j x_{kj} = 1, \\
& \Gamma_{kj} \ = \sum_{i=1, i\neq k}^{n} \gamma_{ij}, \\
& \Phi_{kr} \ = \sum_{i=1, i\neq k}^{n} \phi_{ir}, \\
& \Gamma_{kj} \leq v_j x_{kj} - \epsilon, \ j = 1, \ldots, c, \\
& \phi_{ir} \leq u_r y_{ir} - \epsilon, \ r = 1, \ldots, d; \ i = 1, \ldots, n; \ i \neq k, \\
& v_j, u_r, \Delta_i^O, \Delta_i^I, \Gamma_{kj}, \gamma_{ij}, \Phi_{kr}, \phi_{ir}, \epsilon \geq 0; \ i = 1, \ldots, \ n; \ j = 1, \ldots, \ m; \ r = 1, \ldots, \ s.
\end{aligned}
$$

The variables Δ_i^O, Δ_i^I can be eliminated with the following steps:

By the constraints of $(M4)$,

$$\Delta_i^O, \Delta_i^I \geq 0,$$

which means

$$\sum_{r=1}^{s} u_r y_{ir} - \sum_{j=1}^{m} v_j x_{ij} + \Delta_i^O + \Delta_i^I - \sum_{r=1}^{d} \phi_{ir} - \sum_{j=1}^{c} \gamma_{kj} = 0,\ i = 1, \ldots, n;\ i \neq k,$$

can be rewritten as

$$\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} + \sum_{j=1}^{c} \gamma_{kj} \geq 0,\ i = 1, \ldots, n;\ i \neq k. \tag{2.13}$$

Also by the constraints of $(M4)$, since

$$\Delta_i^I + \Delta_i^O = \sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} + \sum_{j=1}^{c} \gamma_{kj},$$

the objective function

$$Min \sum_{i=1, i\neq k}^{n} (\Delta_i^O + \Delta_i^I),$$

can be rewritten as

$$Min \sum_{i=1, i\neq k}^{n} \left(\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} + \sum_{j=1}^{c} \gamma_{kj} \right).$$

Also by $(M4)$

$$\Gamma_{kj} = \sum_{i=1, i\neq k}^{n} \gamma_{ij},\ \Phi_{kr} = \sum_{i=1, i\neq k}^{n} \phi_{ir}, \text{ and} \sum_{r=1}^{d} \Phi_{kr} + \sum_{j=1}^{c} \Gamma_{kj} = \sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr}.$$

Thus, the new objective function is

$$Min \sum_{i=1}^{n} \left(\sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} \right). \tag{2.14}$$

Incorporating Eqns. (2.13) and (2.14) into $(M4)$, the model $(M4)$ can be re-written as the following equivalent LP model $(M5)$. The new objective function is defined by equation (2.14), and equation (2.13) gives us the second constraint of the following model $(M5)$.

$$
\begin{aligned}
(M5)\quad & Min \sum_{i=1}^{n}\left(\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir}\right) \\
& \text{subject to} \\
& \sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} - \sum_{r=1}^{d} \Phi_{kr} - \sum_{j=1}^{c} \Gamma_{kj} \;=\; 0, \\
& \sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} + \sum_{j=1}^{c} \gamma_{ij} \geq 0,\; i = 1, \ldots, n;\; i \neq k, \\
& \sum_{j=1}^{m} v_j x_{kj} = 1, \\
& \Gamma_{kj} \;=\; \sum_{i=1, i\neq k}^{n} \gamma_{ij}, \\
& \Phi_{kr} \;=\; \sum_{i=1, i\neq k}^{n} \phi_{ir}, \\
& \Gamma_{kj} \leq v_j x_{kj} - \epsilon,\; j = 1, \ldots, c, \\
& \phi_{ir} \leq u_r y_{ir} - \epsilon,\; r = 1, \ldots, d;\; i = 1, \ldots, n;\; i \neq k, \\
& v_j, u_r, \Gamma_{kj}, \gamma_{ij}, \Phi_{kr}, \phi_{ir}, \epsilon \geq 0;\; i = 1, \ldots,\; n;\; j = 1, \ldots,\; m;\; r = 1, \ldots,\; s.
\end{aligned}
$$

Model $(M5)$ is an LP model. By applying this model any k^{th} DMU under common weights can improve its efficiency by reducing inputs under CSOI, or increasing output under CSOO, or both. The model is feasible since it is the linear form of model $(M3)$, which has already been proved to be feasible. However, model $(M5)$ does not minimize the amount of input/output transfer. Let $\theta_i^{CW*}(i = 1, \ldots, n)$ be the common-weight efficiency of all DMUs after the applying model $(M5)$. We now design a model to minimize the input/output transfer while ensuring that the efficiency of the DMUs does not reduce any further. This LP model $(M6)$ is as follows:

$$
\begin{aligned}
(M6)\quad & Min \sum_{r=1}^{d} \Phi_{kr} + \sum_{j=1}^{c} \Gamma_{kj} \\
& \text{subject to} \\
& \sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} - \sum_{r=1}^{d} \Phi_{kr} - \sum_{j=1}^{c} \Gamma_{kj} \;=\; 0, \\
& \theta_i^{CW*}\left(\sum_{j=1}^{m} v_j x_{ij} + \sum_{j=1}^{c} \gamma_{ij}\right) - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} \leq 0,\; i = 1, \ldots, n;\; i \neq k, \\
& \sum_{j=1}^{m} v_j x_{ij} + \sum_{j=1}^{c} \gamma_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \phi_{ir} \geq 0,\; i = 1, \ldots, n;\; i \neq k \\
& \sum_{j=1}^{m} v_j x_{kj} = 1,
\end{aligned}
$$

$$\Gamma_{kj} = \sum_{i=1, i\neq k}^{n} \gamma_{ij},$$

$$\Phi_{kr} = \sum_{i=1, i\neq k}^{n} \phi_{ir},$$

$$\Gamma_{kj} \leq v_j x_{kj} - \epsilon,\ j = 1, \ldots, c,$$

$$\phi_{ir} \leq u_r y_{ir} - \epsilon;\ r = 1, \ldots, d;\ i = 1, \ldots, n;\ i \neq k,$$

$$v_j, u_r, \Gamma_{kj}, \gamma_{ij}, \Phi_{kr}, \phi_{ir}, \epsilon \geq 0;\ i = 1, \ldots,\ n;\ j = 1, \ldots,\ m;\ r = 1, \ldots,\ s.$$

Applying model ($M5$), followed by model ($M6$), we can calculate the minimum necessary input/output change that allows the observed DMU k to become efficient, without reducing overall efficiency in the system.

3 Numerical Example

The example uses data for 14 hospitals, with the inputs being the number of doctors (Input 1), nurses (Input 2) and the outputs being the number of outpatients (Output 1), inpatients (Output 2). The data is obtained from the work of Cooper et al. (2007). In this example, we assume that the number of doctors (Input 1) and inpatients (Output 2) are under the CSOI and CSOO constraint respectively, and that the DMUs are under Common Weights (CW). These assumptions are made only as part of the demonstration. The data for the DMUs, as well as the original efficiency score under CW, is shown in the table below.

Table 1: Input, Output, and Common Weight Efficiency Data of 14 Hospitals

Hospital (DMU no.)	Input 1 Doctors	Input 2 Nurses	Output 1 Outpatients	Output 2 Inpatients	CW Efficiency θ_i^{CW}
1	3008	20980	97775	101225	0.759
2	3985	25643	135871	130580	0.799
3	4324	26978	133655	168473	0.980
4	3534	25361	46243	100407	0.623
5	8836	40796	176661	215616	0.821
6	5376	37562	182576	217615	0.911
7	4982	33088	98880	167278	0.794
8	4775	39122	136701	193393	0.781
9	8046	42958	225138	256575	0.933
10	8554	48955	257370	312877	1.000
11	6147	45514	165274	227099	0.786
12	8366	55140	203989	321623	0.916
13	13479	68037	174270	341743	0.783
14	21808	78302	322990	487539	0.957
				Average Eff.	0.846

We choose DMU 4 for the target of efficiency improvement, since it has the lowest CW-efficiency. Setting $k = 4$, we apply model ($M5$) to the data. The intermediate constant sum constraint input/output values resulting from ($M5$) are shown in table 2. Using the results of ($M5$), we apply model ($M6$) to the data. The new values after the redistribution is shown in table 3. Table 3 shows that after the redistribution, the overall efficiency of the system is better than before, even if individual efficiencies may be less, and the observed DMU has efficiency of 1.

Table 2: Intermediate Input/Output Values Under Constant Sum Constraint.

DMU no.	Intermediate Input 1	Intermediate Output 2	Intermediate eff.
1	3007.99	602.89	0.898
2	3985.00	3091.39	0.998
3	4980.34	168473.00	1.000
4	0.00	431609.48	1.000
5	8836.00	215616.00	0.821
6	5376.00	114524.11	1.000
7	4982.00	167278.00	0.671
8	4775.00	193393.00	0.811
9	8970.22	256574.56	1.000
10	10507.45	312876.56	1.000
11	6147.00	227099.00	0.817
12	8366.00	321623.00	0.820
13	13479.00	341743.00	0.538
14	21808.00	487539.00	0.724
		Average eff.	0.864

Table 3: Final Results of Input/Output Redistribution After (M6)

DMU no.	Old Input 1	Old Output 2	Old CW Eff.	New Input 1	New Output 2	New eff.
1	3008	101225	0.759	3008.0	101225.0	0.941
2	3985	130580	0.799	3985.0	130580.0	0.998
3	4324	168473	0.980	4835.2	158890.1	1.000
4	3534	100407	0.623	979.1	135711.7	1.000
5	8836	215616	0.821	8836.0	215616.0	0.821
6	5376	217615	0.911	5760.7	203263.0	1.000
7	4982	167278	0.794	4982.0	167278.0	0.807
8	4775	193393	0.781	4775.0	193393.0	0.898
9	8046	256575	0.933	8046.0	250721.4	1.000
10	8554	312877	1.000	10213.1	307360.8	1.000
11	6147	227099	0.786	6147.0	227099.0	0.887
12	8366	321623	0.916	8366.0	321623.0	0.950
13	13479	341743	0.783	13479.0	341743.0	0.691
14	21808	487539	0.957	21808.0	487539.0	0.796
		Average eff.	0.846			0.914

4 Conclusion

In this paper, we have developed DEA models to calculate a strategy for decreasing inputs or increasing outputs until an inefficient DMU becomes efficient, when there is a constant sum of inputs/outputs, and the efficiency of all DMUs is measured using a common set of weights. While there exists work on the common set of weights problem in DEA, and the constant sum of inputs/outputs problem, this paper's models address situation where both constraints are in operation. The problem is solved with a two-step process using two LP models. The first model determines the necessary amount of input/output change in the observed DMU which leads to it achieving efficiency. The second LP model determines the minimum change in each parameter while ensuring that overall efficiency in the system does not suffer. These models address a gap in existing literature, and can be applied in any situation where a single DMU is seeking to improve efficiency when there is a limit on total input/output, and parameter weights have a common value. It may be noted that while the observed DMU's efficiency and the overall efficiency has been improved, some individual DMUs may suffer efficiency reduction. A future direction of research may be to improve the observed DMU and overall efficiency, without reducing efficiency in any other DMU.

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